

Conifold Transitions for Complete Intersection Calabi-Yau 3–folds in Products of Projective Spaces

Jinxing Xu

School of Mathematical Sciences, University of Science and Technology

Hefei, 230026, P. R. China

E-mail: xujx02@ustc.edu.cn

Abstract

We prove that a generic complete intersection Calabi-Yau 3–fold defined by sections of ample line bundles on a product of projective spaces admits a conifold transition to a connected sum of $S^3 \times S^3$. In this manner, we obtain complex structures with trivial canonical bundles on some connected sums of $S^3 \times S^3$. This construction is an analogue of that made by Friedman, Lu and Tian who used quintics in \mathbb{P}^4 .

Keywords Calabi-Yau threefolds, conifold transitions, complex structures on connected sums of $S^3 \times S^3$

MR(2000) Subject Classification 14J32

1 Introduction

In this paper, we deal with conifold transitions for some complete intersection Calabi-Yau 3–folds in products of projective spaces. Let us first recall some notions.

Definition 1 ([11]). *Let Y be a Calabi-Yau threefold and $\phi : Y \rightarrow \bar{Y}$ be a birational contraction onto a normal variety. If there exists a complex deformation (smoothing) of \bar{Y} to a smooth complex threefold \tilde{Y} , then the process of going from Y to \tilde{Y} is called a geometric transition and denoted by $T(Y, \bar{Y}, \tilde{Y})$. A transition $T(Y, \bar{Y}, \tilde{Y})$ is called conifold if \bar{Y} admits only ordinary double points as singularities and the resolution morphism ϕ is a small resolution (i.e. replacing each ordinary double point by a smooth rational curve).*

Note for a conifold transition $T(Y, \bar{Y}, \tilde{Y})$, the exceptional set of the morphism ϕ is several pairwise disjoint smooth rational curves each with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ in Y and conversely, given a finite set of pairwise disjoint smooth rational curves each with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ in Y , we can contract them to get \bar{Y} admitting only ordinary double points as singularities. The smoothing of \bar{Y} has been studied by several people. For example, we have the following theorem:

Theorem 1. *Let \bar{Y} be a singular threefold with l ordinary double points as the only singular points p_1, \dots, p_l . Let Y be a small resolution of \bar{Y} by replacing p_i by smooth rational curves C_i . Assume that Y is cohomologically Kähler and has trivial canonical line bundle. Furthermore, we assume that the fundamental classes $[C_i]$ in $H^2(Y; \Omega_Y^2)$ satisfy a relation $\sum_i \lambda_i [C_i] = 0$ with each λ_i nonzero. Then \bar{Y} can be deformed into a smooth threefold \tilde{Y} .*

The above theorem is taken from [12]. Y. Kawamata has proven similar results. A special case of the above theorem was obtained by R. Friedman in [3].

The conifold transition process was firstly (locally) observed by H. Clemens in [2], where he explained that locally a conifold transition is described by a suitable $S^3 \times D_3$ to $S^2 \times D_4$ surgery. Roughly speaking, the conifold transition $T(Y, \bar{Y}, \tilde{Y})$ from Y to \tilde{Y} kills 2-cycles in Y and increases 3-cycles in \tilde{Y} . For a precise relation between their Betti numbers, one can consult Theorem 3.2 in [11]. In Theorem 1, if the fundamental classes $[C_i]$ generates $H^4(Y; \mathbb{C})$, then we would have $b_2(\tilde{Y}) = 0$. By results of C.T.C. Wall in [15], \tilde{Y} would be diffeomorphic to a connected sum of $S^3 \times S^3$, and the number of copies is $\frac{b_3(\tilde{Y})}{2} + l - b_2(Y)$. The precise statement of a particular case of Wall's result is:

Theorem 2. ([4]) *Suppose M is a simply connected compact differential manifold with dimension 6. If $H_2(M; \mathbb{Z}) = 0$, and $H_3(M; \mathbb{Z})$ is torsion free. Then M is diffeomorphic to a connected sum of $S^3 \times S^3$.*

R. Friedman [4], Lu and Tian [9] considered conifold transitions for quintics in \mathbb{P}^4 . Using Theorems 1 and 2, they obtained complex structures with trivial canonical bundles on the connected sum of m copies of $S^3 \times S^3$ for each $m \geq 2$. In view of the following Corollary 1, we also obtained complex structures with trivial canonical bundles on some connected sums of $S^3 \times S^3$.

In this paper, we will prove that a generic complete intersection Calabi-Yau 3-fold defined by sections of ample line bundles in a product of projective spaces also admits a conifold transition to a connected sum of $S^3 \times S^3$.

Let $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ be a product of projective spaces of dimension $\sum_{i=1}^k n_i = m + 3$, with $m \geq 1$.

Take ample line bundles on X : $L_i = \pi_1^* \mathcal{O}_{\mathbb{P}^{n_1}}(d_1^{(i)}) \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \pi_k^* \mathcal{O}_{\mathbb{P}^{n_k}}(d_k^{(i)})$, where $\pi_j : X \rightarrow \mathbb{P}^{n_j}$ is the natural projection, and $d_j^{(i)} > 0$, $\forall 1 \leq j \leq k, 1 \leq i \leq m$.

In this case, by Bertini's theorem, we know that the complete intersection Y defined by generic sections $s_i \in H^0(X, L_i) (1 \leq i \leq m)$: $Y = \{p \in X : s_i(p) = 0, \forall 1 \leq i \leq m\}$ is a smooth subscheme of X , and $\dim Y = 3$. If the line bundles satisfy the Calabi-Yau condition:

$$\sum_{i=1}^m d_j^{(i)} = n_j + 1, \forall j = 1, \dots, k.$$

Then Y is a Calabi-Yau 3-fold. And in this case we call Y is a complete intersection Calabi-Yau 3-fold (CICY) in X with configuration matrix:

$$D = \begin{pmatrix} d_1^{(1)} & \cdots & d_k^{(1)} \\ \vdots & & \vdots \\ d_1^{(m)} & \cdots & d_k^{(m)} \end{pmatrix}$$

Since $H_2(X; \mathbb{Z}) = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_k$ is a free \mathbb{Z} -module with rank k , where e_i is a generator of $H_2(\mathbb{P}^{n_i}; \mathbb{Z}) \simeq \mathbb{Z}$, for $1 \leq i \leq k$. We call a smooth rational curve C in X has degree (d_1, \dots, d_k) if the fundamental class of C in X satisfies: $[C] = d_1 e_1 + \cdots + d_k e_k$.

Our main result is the following:

Theorem 3. *Let $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ be a product of projective spaces of dimension $\sum_{i=1}^k n_i = m + 3$, with $m \geq 1$. Fix a configuration matrix*

$$D = \begin{pmatrix} d_1^{(1)} & \cdots & d_k^{(1)} \\ \vdots & & \vdots \\ d_1^{(m)} & \cdots & d_k^{(m)} \end{pmatrix}$$

with $d_j^{(i)} > 0 (\forall 1 \leq j \leq k, 1 \leq i \leq m)$ and satisfying the Calabi-Yau condition $\sum_{i=1}^m d_j^{(i)} = n_j + 1 (\forall j = 1, \dots, k)$, then a generic complete intersection Calabi-Yau 3-fold Y in X with configuration matrix D contains $k + 1$ pairwise disjoint smooth rational curve $C_i (1 \leq i \leq k + 1)$ such that the normal bundle $N_{C_i, Y} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) (\forall 1 \leq i \leq k + 1)$, the degree of C_i is $(0, \dots, 0, 1, 0, \dots, 0)$, where 1 is at the i -th place, $\forall 1 \leq i \leq k$, and the degree of C_{k+1} is $(1, \dots, 1)$.

Corollary 1. *For a generic complete intersection Calabi-Yau 3-fold Y as in Theorem 3, Y admits a conifold transition to a connected sum of $S^3 \times S^3$.*

Proof of the Corollary:

We verify the $k+1$ curves in Theorem 3 satisfy the conditions in Theorem 1. Since Y is a complete intersection by sections of very ample line bundles, Lefschetz's hyperplane theorem implies $H_2(Y; \mathbb{Z}) \simeq H_2(X; \mathbb{Z})$. By Poincaré Duality, $H_2(Y; \mathbb{Z}) \simeq H^4(Y; \mathbb{Z})$, then it is easy to verify the cohomology relation in Theorem 1 is satisfied by the $k+1$ rational curves in Theorem 3. So we can apply Theorem 1.

Using the analysis of homology groups of Y and \tilde{Y} in [11], one can show that after the conifold transition of Y , the manifold \tilde{Y} satisfies the hypothesis in Theorem 2. This finishes the proof of the corollary. \square

In [5], P.S.Green and T.Hübsch proved that the moduli spaces of complete intersection Calabi-Yau 3-folds in products of projective spaces were connected each other by conifold transitions. But their results do not imply that a generic complete intersection Calabi-Yau 3-fold admits a conifold transition to a connected sum of $S^3 \times S^3$. In [14], C. Voisin proved that, among other results, that for a Calabi-Yau 3-fold, the integral homology group is generated by the fundamental classes of curves in it. In [7] and [8], the authors studied the existence of isolated smooth rational curves in a generic complete intersection Calabi-Yau 3-fold in a projective space. So a natural problem is that whether their results can be generalized to complete intersection Calabi-Yau 3-folds in a product of projective spaces. Our results can be viewed as an attempt towards that problem.

The paper is organized as follows:

In section 2, we prove a general proposition for the existence of isolated smooth rational curves in a family of Calabi-Yau 3-folds.

In section 3, we recall a deformation proposition about isolated smooth rational curves in Calabi-Yau 3-folds.

In section 4, we combine the results in the preceding two sections and specialize them to the case of complete intersection Calabi-Yau 3-folds in a product of projective spaces, then we get the existence of $k+1$ isolated smooth rational curves as in Theorem 3.

In section 5, we analyze the dimension of some incident variety and prove that the $k+1$ smooth rational curves in Theorem 3 are pairwise disjoint, then this would complete the proof of Theorem 3.

Acknowledgements: The author would like to express sincere thanks to his thesis advisor Professor Gang Tian for introducing him to this problem and for his continuous encouragement.

2 A general lemma for existence of isolated rational curves

In this section, we consider a complex projective smooth variety X and define two closed subvarieties $C \subset Y$ by complete intersections of sections of line bundles. Then in some cases, we will compute the normal bundle $N_{C,Y}$ of C in Y .

Suppose X is a complex projective smooth variety with dimension $m + 3$, $m \geq 1$. $L_i (1 \leq i \leq m)$ and $\tilde{L}_j (1 \leq j \leq m + 2)$ are invertible sheaves on X . Given non-zero sections $\tilde{s}_j \in H^0(X, \tilde{L}_j)$, for $1 \leq j \leq m + 2$. Then these sections generate a sheaf of ideal $I_C = \sum_{j=1}^{m+2} \mathcal{O}_X \tilde{s}_j$ on X : locally $I_C = \sum_{j=1}^{m+2} \mathcal{O}_X \tilde{f}_j$, where $\tilde{s}_j = \tilde{f}_j \tilde{e}_j$ and \tilde{e}_j is a local frame of the invertible sheaf \tilde{L}_j , $1 \leq j \leq m + 2$. Suppose the closed subscheme C defined by I_C is a smooth rational curve: $C \simeq \mathbb{P}^1$ and moreover, the $m + 2$ sections $\tilde{s}_j (1 \leq j \leq m + 2)$ is a regular sequence at each point of C , that is, for any closed point $p \in C$, if around p , $\tilde{s}_j = \tilde{f}_j \tilde{e}_j$ and \tilde{e}_j is a local frame of the invertible sheaf \tilde{L}_j , then the local regular functions $\tilde{f}_j (1 \leq j \leq m + 2)$ constitute a regular sequence at p .

Let $L_{ji} = L_i \otimes_{\mathcal{O}_X} \tilde{L}_j^{-1}$, and suppose $d_{ji} = \deg L_{ji}|_C \geq 0$, $\forall 1 \leq i \leq m, 1 \leq j \leq m + 2$. Given sections $s_{ji} \in H^0(X, L_{ji})$, we get sections $s_i = \sum_{j=1}^{m+2} \tilde{s}_j \otimes s_{ji} \in H^0(X, L_i)$, $\forall 1 \leq i \leq m, 1 \leq j \leq m + 2$. Similarly as for I_C , the sections s_i also generate a sheaf of ideal $I_Y = \sum_{i=1}^m \mathcal{O}_X s_i$, and I_Y determines a closed subscheme Y of X . By the definition of s_i , we have $I_Y \subset I_C$ and $C \subset Y$.

Now fix a linear subspace $V_{ji} \subset H^0(X, L_{ji})$, for each $1 \leq i \leq m, 1 \leq j \leq m + 2$ and suppose they satisfy the following three conditions:

- For any $f_{ji} \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d_{ji}))$, there exists a $s_{ji} \in V_{ji}$, such that $s_{ji}|_C = f_{ji}$, $\forall 1 \leq i \leq m, 1 \leq j \leq m + 2$.
- For generic choice of $s_{ji} \in V_{ji}, 1 \leq i \leq m, 1 \leq j \leq m + 2$, the m sections $s_i = \sum_{j=1}^{m+2} \tilde{s}_j \otimes s_{ji} \in H^0(X, L_i) (1 \leq i \leq m)$ is a regular sequence at each point of C , and C is located in the smooth locus of the subscheme Y defined by the sheaf of ideal $I_Y = \sum_{i=1}^m \mathcal{O}_X s_i$.
- $\sum_{j=1}^{m+2} \deg \tilde{L}_j|_C = \sum_{i=1}^m \deg L_i|_C - 2$.

Under the above hypothesis, we have the following proposition:

Proposition 1. *For generic choices of $s_{ji} \in V_{ji}$, $1 \leq i \leq m$, $1 \leq j \leq m+2$, the normal bundle $N_{C,Y} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.*

Proof. In general, if X_1 is a complex smooth subvariety located in the smooth locus of a complex variety X_2 , and the sheaf of ideal of X_2 defining X_1 is I_{X_1} , then we have $\frac{I_{X_1}}{I_{X_1}^2}|_{X_1} \simeq N_{X_1,X_2}^*$, where $N_{X_1,X_2}^* = \mathcal{H}om_{\mathcal{O}_{X_1}}(N_{X_1,X_2}, \mathcal{O}_{X_1})$ is the conormal bundle of X_1 in X_2 .

Now $C \subset Y \subset X$, C, X are smooth and C is located in the smooth locus of Y , so we have the following exact sequence:

$$N_{Y,X}^*|_C \rightarrow N_{C,X}^* \rightarrow N_{C,Y}^* \rightarrow 0$$

On the other hand, we have the following isomorphisms:

$$\begin{aligned} \oplus_{i=1}^m L_i^*|_C &\xrightarrow{\varphi|_C} \frac{I_{Y,X}}{I_{Y,X}^2}|_C \simeq N_{Y,X}^*|_C \\ \oplus_{j=1}^{m+2} \tilde{L}_j^*|_C &\xrightarrow{\tilde{\varphi}} \frac{I_{C,X}}{I_{C,X}^2} \simeq N_{C,X}^* \end{aligned}$$

where $\varphi : \oplus_{i=1}^m L_i^*|_Y \rightarrow \frac{I_{Y,X}}{I_{Y,X}^2}$ and $\tilde{\varphi} : \oplus_{j=1}^{m+2} \tilde{L}_j^*|_C \rightarrow \frac{I_{C,X}}{I_{C,X}^2}$ are defined as follows:

$$\varphi(e_i^*|_Y) = f_i, \forall 1 \leq i \leq m, \text{ where } e_i \text{ is a local frame of } L_i \text{ and } s_i = f_i e_i.$$

$$\tilde{\varphi}(\tilde{e}_j^*|_C) = \tilde{f}_j, \forall 1 \leq j \leq m+2, \text{ where } \tilde{e}_j \text{ is a local frame of } \tilde{L}_j \text{ and } \tilde{s}_j = \tilde{f}_j \tilde{e}_j.$$

It is easy to verify that the homomorphisms φ and $\tilde{\varphi}$ are well-defined and both are surjective. Since $\frac{I_{Y,X}}{I_{Y,X}^2}|_C$ and $\oplus_{i=1}^m L_i^*|_C$ are locally free sheaves on C with the same rank (here we have used the hypothesis that $\{s_i\}_{i=1}^m$ is a regular sequence at each point of C), we get that $\varphi|_C$ is an isomorphism. Similarly, $\tilde{\varphi}$ is an isomorphism.

In view of the above exact sequence and the isomorphisms, we get the following exact sequence of locally free sheaves on C :

$$\oplus_{i=1}^m L_i^*|_C \rightarrow \oplus_{j=1}^{m+2} \tilde{L}_j^*|_C \rightarrow N_{C,Y}^* \rightarrow 0$$

Applying $\mathcal{H}om_{\mathcal{O}_C}(\cdot, \mathcal{O}_C)$ to this exact sequence, we get the exact sequence

$$0 \rightarrow N_{C,Y} \rightarrow \oplus_{j=1}^{m+2} \tilde{L}_j|_C \xrightarrow{\psi} \oplus_{i=1}^m L_i|_C \quad (1)$$

The homomorphism ψ is determined as following:

$$\begin{aligned} \pi_i \circ \psi \circ \iota_j : \tilde{L}_j|_C &\rightarrow L_i|_C \\ \tilde{e}_j|_C &\rightarrow s_{ji}|_C \otimes \tilde{e}_j|_C \end{aligned}$$

where $\iota_j : \tilde{L}_j|_C \rightarrow \oplus_{j=1}^{m+2} \tilde{L}_j^*|_C$ is the natural inclusion, $\pi_i : \oplus_{i=1}^m L_i^*|_C \rightarrow L_i|_C$ is the natural projection, and \tilde{e}_j is a local frame of \tilde{L}_j .

Finally, Proposition 1 is a consequence of the following lemma applied to the exact sequence (1).

□

Lemma 1. *Suppose $M = \oplus_{i=1}^m \mathbb{C}[s, t]e_i$, $N = \oplus_{j=1}^{m+2} \mathbb{C}[s, t]\tilde{e}_j$ are graded free $\mathbb{C}[s, t]$ -modules with rank m and $m + 2$ respectively. Suppose $\sum_{j=1}^{m+2} \deg \tilde{e}_j = \sum_{i=1}^m \deg e_i + 2$ and $d_{ji} = \deg \tilde{e}_j - \deg e_i \geq 0$, $\forall 1 \leq i \leq m, 1 \leq j \leq m + 2$. Then the kernel of a generic homomorphism $\varphi : N \rightarrow M$ as graded $\mathbb{C}[s, t]$ -modules is a free $\mathbb{C}[s, t]$ -module and $\text{Ker} \varphi \simeq \mathbb{C}[s, t]e'_1 \oplus \mathbb{C}[s, t]e'_2$, with $\deg e'_1 = \deg e'_2 = 1$.*

Here generic means that if we write $\varphi(\tilde{e}_j) = \sum_{i=1}^m f_{ji}e_i$, where f_{ji} is a homogenous polynomial of s, t with degree d_{ji} , then the coefficients of f_{ji} are chosen generically, $\forall 1 \leq i \leq m, 1 \leq j \leq m + 2$.

Proof. Taken homogenous polynomials f_{ji} of s, t with degree d_{ji} , $1 \leq i \leq m, 1 \leq j \leq m + 2$. Then we get the associated homomorphism $\varphi : N \rightarrow M$, $\varphi(\tilde{e}_j) = \sum_{i=1}^m f_{ji}e_i$. Denote the kernel $\text{Ker} \varphi$ as L , then as a vector space over \mathbb{C} , we have $L = \oplus_{l \in \mathbb{Z}} L_l$, where L_l is the degree l part of L and by definition,

$$L_l = \left\{ \sum_{j=1}^{m+2} g_j \tilde{e}_j \mid g_j \text{ is a homogenous polynomial of } s, t \text{ with degree } d_j, \right. \\ \left. d_j + \deg \tilde{e}_j = l, \forall 1 \leq j \leq m + 2. \sum_{j=1}^{m+2} f_{ji} g_j = 0, \forall 1 \leq i \leq m. \right\}$$

So in order to prove this lemma, it suffices to prove that, for generic choices of the homogenous polynomials f_{ji} :

$$\dim_{\mathbb{C}} L_l = 0, \text{ if } l \leq 0;$$

$$\dim_{\mathbb{C}} L_l = 2l, \text{ if } l \geq 0.$$

Now we analyze the linear space L_l .

Since $\deg g_j = d_j \geq 0$, $\deg f_{ji} = d_{ji} \geq 0$, we can write:

$$g_j = \sum_{k=0}^{d_j} b_j^{(k)} s^k t^{d_j-k}$$

$$f_{ji} = \sum_{k=0}^{d_{ji}} a_{ji}^{(k)} s^k t^{d_{ji}-k}$$

where $b_j^{(k)}, a_{ji}^{(k)}$ are coefficients of g_j, f_{ji} respectively.

Then

$$f_{ji}g_j = \begin{pmatrix} t^{d_j+d_{ji}}, & t^{d_j+d_{ji}-1}s, & \dots, & s^{d_j+d_{ji}} \end{pmatrix} A_{ji} \begin{pmatrix} b_j^{(0)} \\ \vdots \\ b_j^{(d_j)} \end{pmatrix}$$

where the matrix A_{ji} is:

$$A_{ji} = \begin{pmatrix} a_{ji}^{(0)} & & & & \\ a_{ji}^{(1)} & a_{ji}^{(0)} & & & \\ \vdots & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & a_{ji}^{(0)} & \\ a_{ji}^{(d_{ji})} & \ddots & \ddots & \vdots & \\ & \ddots & \ddots & \vdots & \\ & & \ddots & \vdots & \\ & & & a_{ji}^{(d_{ji})} \end{pmatrix}$$

Then it is easy to see that L_l can be identified with the space of solutions of the following system of linear equations for $b_j^{(k)}$:

$$A \cdot \begin{pmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vdots \\ \vec{b}_{m+2} \end{pmatrix}$$

where

$$A = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{m+2,1} \\ \vdots & \vdots & & \vdots \\ A_{1m} & A_{2m} & \cdots & A_{m+2,m} \end{pmatrix}, \vec{b}_j = \begin{pmatrix} b_j^{(0)} \\ \vdots \\ b_j^{(d_j)} \end{pmatrix}$$

It is then an elementary exercise to show that the above matrix A has full rank for generic polynomials f_{ji} , $1 \leq i \leq m, 1 \leq j \leq m+2$. So the dimension of its solution space is equal to the difference of the number of its columns and the number of its rows.

The number of columns of the matrix A is :

$$\sum_{j=1}^{m+2} d_j + m + 2$$

The number of rows of the matrix A is:

$$\sum_{i=1}^m (d_j + d_{ji}) + m$$

Recall $d_j + \deg \tilde{e}_j = l$, $d_{ji} = \deg \tilde{e}_j - \deg e_i$, so $d_j + d_{ji} = l - \deg e_i$, $\forall 1 \leq i \leq m, 1 \leq j \leq m+2$. Then:

$$\begin{aligned} & \text{the number of columns of } A - \text{the number of rows of } A \\ &= \sum_{j=1}^{m+2} d_j + m + 2 - \left(\sum_{i=1}^m (d_j + d_{ji}) + m \right) \\ &= - \sum_{j=1}^{m+2} \deg \tilde{e}_j + (m+2)l + m + 2 - \left(- \sum_{i=1}^m \deg e_i + ml + m \right) \\ &= 2l \end{aligned}$$

This is just what we want. So we have proven this lemma. \square

Remark 1. In the appendix of [6], S. Katz computed the normal bundles of rational curves with $d \leq 3$ on a quintic 3-fold. Our proof of Proposition 1 is motivated by the computations there.

3 A deformation property

In this section, we will prove that for a flat family of varieties, if a member of this family contains a smooth rational curve with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, then a generic member of this family also contains such a rational curve. This proposition is a direct consequence of infinitesimal properties of Hilbert schemes and is more or less well known, so we will just state the proposition in a form that we will use and briefly indicate the proof.

Proposition 2. Suppose X is a smooth projective complex variety with dimension $m+3$, $m \geq 1$. L_i ($1 \leq i \leq m$) are line bundles over X , and $s'_i \in H^0(X, L_i)$ is a section of L_i , for each $1 \leq i \leq m$. Let Y_0 be the subscheme defined by the sheaf of ideal $I_{Y_0} = \sum_{i=1}^m \mathcal{O}_X s'_i$. Suppose that there is a smooth rational curve C_0 contained in the smooth locus of Y_0 . $\{s'_i\}_{i=1}^m$ is a regular sequence at each point of C_0 and the normal bundle $N_{C_0, Y_0} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.

Then for a generic section $s_i \in H^0(X, L_i)$ ($1 \leq i \leq m$), the subscheme Y defined by the sheaf of ideal $I_Y = \sum_{i=1}^m \mathcal{O}_X s_i$ contains a smooth rational curve C in its smooth locus and the normal bundle $N_{C, Y} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.

Proof. Let $S = H^0(X, L_1) \times \cdots \times H^0(X, L_m)$ be the parametrization space for sections of $L_i (1 \leq i \leq m)$. Consider the following universal family:

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{i} & X \times S \\ \pi \downarrow & & \downarrow \pi_2 \\ S & \xlongequal{\quad} & S \end{array}$$

where π_2 is the natural projection to S , and $i : \mathcal{Y} \rightarrow X \times S$ is a closed embedding, defined by a sheaf of ideal $I_{\mathcal{Y}}$ such that at each point p of $X \times S$, if $\pi_2(p) = (s_1, \dots, s_m) \in S$, then $I_{\mathcal{Y}}$ is generated by s_1, \dots, s_m at p . (Let $\pi_1 : X \times S \rightarrow X$ be the natural projection, then after the pulled back by π_1 , s_i can be viewed as a section of $\pi_1^* L_i$.)

Consider the open subscheme \mathcal{U} of \mathcal{Y} such that $\pi|_{\mathcal{U}} : \mathcal{U} \rightarrow S$ is a smooth morphism. By the hypothesis, $C_0 \subset \mathcal{U}$. Then by Corollary (2.14) of [13] and the fact that $H^1(\mathbb{P}^1, T_{\mathbb{P}^1}) = 0$, we know that for a generic point p of S , the fiber $\mathcal{U}_p = \pi|_{\mathcal{U}}^{-1}(p)$ contains a smooth rational curve C_p . Since $\text{Ext}_{\mathbb{P}^1}^1(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) = 0$, the rank 2 bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ on \mathbb{P}^1 is infinitesimally rigid, so we get that for a generic point p of S , the normal bundle $N_{C_p, \mathcal{U}_p} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. \square

4 Existence of rational curves in CICYs

Now we specialize the results in the preceding sections to Calabi-Yau 3-folds in a product of projective spaces. We will construct line bundles L_i , \tilde{L}_j , subspaces V_{ji} of $H^0(X, L_{ji})$ and sections \tilde{s}_j of \tilde{L}_j as in section 2. Then we show these data satisfy the conditions of Proposition 1.

Let $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ be a product of projective spaces of dimension $\sum_{i=1}^k n_i = m + 3$, with $m \geq 1$.

Take line bundles on X : $L_i = \pi_1^* \mathcal{O}_{\mathbb{P}^{n_1}}(d_1^{(i)}) \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \pi_k^* \mathcal{O}_{\mathbb{P}^{n_k}}(d_k^{(i)})$, where $\pi_j : X \rightarrow \mathbb{P}^{n_j}$ is the natural projection, and $d_j^{(i)} > 0$, $\forall 1 \leq j \leq k, 1 \leq i \leq m$.

Suppose the line bundles satisfy the Calabi-Yau condition:

$$\sum_{i=1}^m d_j^{(i)} = n_j + 1, \forall j = 1, \dots, k.$$

Then as we shown in section 1, generic sections $s_i \in H^0(X, L_i) (1 \leq i \leq m)$ define

a complete intersection Calabi-Yau 3-fold (CICY) in X with configuration matrix:

$$D = \begin{pmatrix} d_1^{(1)} & \cdots & d_k^{(1)} \\ \vdots & & \vdots \\ d_1^{(m)} & \cdots & d_k^{(m)} \end{pmatrix}$$

Denote the homogenous coordinates of \mathbb{P}^{n_j} by $(X_{j0}, \dots, X_{jn_j})$, $j = 1, \dots, k$. Patch these coordinates together we get the homogenous coordinates on X :

$$(X_{10}, \dots, X_{1n_1}; X_{20}, \dots, X_{2n_2}; \dots; X_{k0}, \dots, X_{kn_k})$$

Consider the following smooth rational curve C in X with degree $(1, 0, \dots, 0)$:

$$\begin{aligned} \mathbb{P}^1 &\rightarrow X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \\ (s, t) &\rightarrow (s, t, 0, \dots, 0; 1, 0, \dots, 0; \dots; 1, 0, \dots, 0) \end{aligned}$$

As in section 2, this curve is defined by the following sections of line bundles:

$$\begin{aligned} \tilde{L}_1 &= \pi_1^* \mathcal{O}(1), \tilde{s}_1 = X_{12} \in H^0(X, \tilde{L}_1) \\ \tilde{L}_2 &= \pi_1^* \mathcal{O}(1), \tilde{s}_2 = X_{13} \in H^0(X, \tilde{L}_2) \\ &\vdots \\ \tilde{L}_{n_1-1} &= \pi_1^* \mathcal{O}(1), \tilde{s}_{n_1-1} = X_{1n_1} \in H^0(X, \tilde{L}_{n_1-1}) \\ \tilde{L}_{n_1} &= \pi_2^* \mathcal{O}(1), \tilde{s}_{n_1} = X_{21} \in H^0(X, \tilde{L}_{n_1}) \\ &\vdots \\ \tilde{L}_{n_1+n_2-1} &= \pi_2^* \mathcal{O}(1), \tilde{s}_{n_1+n_2-1} = X_{2n_2} \in H^0(X, \tilde{L}_{n_1+n_2-1}) \\ &\vdots \\ \tilde{L}_{m-n_k+1} &= \pi_k^* \mathcal{O}(1), \tilde{s}_{m-n_k+1} = X_{k1} \in H^0(X, \tilde{L}_{m-n_k+1}) \\ &\vdots \\ \tilde{L}_m &= \pi_k^* \mathcal{O}(1), \tilde{s}_m = X_{kn_k} \in H^0(X, \tilde{L}_m) \end{aligned}$$

Now we define the linear subspaces V_{ji} of $H^0(X, L_{ji})$, where $L_{ji} = L_i \otimes_{\mathcal{O}_X} \tilde{L}_j^{-1}$, for $1 \leq i \leq m, 1 \leq j \leq m+2$. Note that $d_{ji} = \deg L_{ji}|_C \geq 0$.

For each $1 \leq i \leq m$, we define the the linear subspaces V_{ji} of $H^0(X, L_{ji})$ in the following way:

For $1 \leq j \leq n_1 - 1$, $V_{ji} = \{f(X_{10}, X_{11})X_{20}^{d_2^{(i)}} \cdots X_{k0}^{d_k^{(i)}} : f(X_{10}, X_{11}) \text{ is a homogenous polynomial of } X_{10}, X_{11} \text{ with degree } d_1^{(i)} - 1.\}$

For $n_1 \leq j \leq n_1 + n_2 - 1$, $V_{ji} = \{f(X_{10}, X_{11})X_{20}^{d_2^{(i)}-1}X_{30}^{d_3^{(i)}} \cdots X_{k0}^{d_k^{(i)}} : f(X_{10}, X_{11}) \text{ is a homogenous polynomial of } X_{10}, X_{11} \text{ with degree } d_1^{(i)}.\}$

\vdots

For $m - n_k + 1 \leq j \leq m$, $V_{ji} = \{f(X_{10}, X_{11})X_{20}^{d_2^{(i)}} \cdots X_{k-1,0}^{d_{k-1}^{(i)}}X_{k0}^{d_k^{(i)}-1} : f(X_{10}, X_{11}) \text{ is a homogenous polynomial of } X_{10}, X_{11} \text{ with degree } d_1^{(i)}.\}$

Then it is direct to verify that all the data we have just defined: the line bundles L_i , \tilde{L}_j , the sections \tilde{s}_j , the rational curve C and the spaces V_{ji} satisfy the hypothesis in proposition 1.

Similarly, for each $1 \leq j \leq k$, we can get a smooth rational curve with degree $(0, \dots, 0, 1, 0, \dots, 0)$, where 1 is at the j -th place, and all the data satisfying the hypothesis in Proposition 1.

As for smooth rational curves with degree $(1, \dots, 1)$, We can also get the data satisfying the hypothesis in Proposition 1. We just give the rational curve C and the sections of line bundles defining C . The subspaces V_{ji} can be defined in a similar way as before.

Define the curve C as

$$\begin{aligned} \mathbb{P}^1 &\rightarrow X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \\ (s, t) &\rightarrow (s, t, 0, \dots, 0; s, t, 0, \dots, 0; \cdots; s, t, 0, \dots, 0) \end{aligned}$$

The sections of line bundles defining C :

$$\begin{aligned} \tilde{L}_1 &= \pi_1^* \mathcal{O}(1) \otimes_{\mathcal{O}_X} \pi_2^* \mathcal{O}(1), \tilde{s}_1 = X_{10}X_{21} - X_{11}X_{20} \in H^0(X, \tilde{L}_1). \\ \tilde{L}_2 &= \pi_1^* \mathcal{O}(1) \otimes_{\mathcal{O}_X} \pi_3^* \mathcal{O}(1), \tilde{s}_2 = X_{10}X_{31} - X_{11}X_{30} \in H^0(X, \tilde{L}_2). \\ &\vdots \\ \tilde{L}_{k-1} &= \pi_1^* \mathcal{O}(1) \otimes_{\mathcal{O}_X} \pi_k^* \mathcal{O}(1), \tilde{s}_{k-1} = X_{10}X_{k1} - X_{11}X_{k0} \in H^0(X, \tilde{L}_{k-1}). \\ \tilde{L}_k &= \pi_1^* \mathcal{O}(1), \tilde{s}_k = X_{12} \in H^0(X, \tilde{L}_k). \\ &\vdots \\ \tilde{L}_m &= \pi_k^* \mathcal{O}(1), \tilde{s}_m = X_{kn_k} \in H^0(X, \tilde{L}_m). \end{aligned}$$

Finally, by Propositions 1 and 2, we get the following:

Proposition 3. *Let $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ be a product of projective spaces of dimension $\sum_{i=1}^k n_i = m + 3$, with $m \geq 1$. Fix a configuration matrix*

$$D = \begin{pmatrix} d_1^{(1)} & \cdots & d_k^{(1)} \\ \vdots & & \vdots \\ d_1^{(m)} & \cdots & d_k^{(m)} \end{pmatrix}$$

with $d_j^{(i)} > 0 (\forall 1 \leq j \leq k, 1 \leq i \leq m)$ and satisfying the Calabi-Yau condition $\sum_{i=1}^m d_j^{(i)} = n_j + 1 (\forall j = 1, \dots, k)$, then a generic complete intersection Calabi-Yau 3-fold Y in X with configuration matrix D contains $k + 1$ smooth rational curves $C_i (1 \leq i \leq k + 1)$ such that the normal bundle $N_{C_i, Y} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) (\forall 1 \leq i \leq k + 1)$, the degree of C_i is $(0, \dots, 0, 1, 0, \dots, 0)$, where 1 is at the i -th place, $\forall 1 \leq i \leq k$, and the degree of C_{k+1} is $(1, \dots, 1)$.

5 Pairwise disjointness of rational curves

In this section, we will show that the $k + 1$ smooth rational curves in Proposition 3 are pairwise disjoint. Let us introduce some notations. We denote $\vec{e}_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{Z}^k$ to be the i -th standard base of \mathbb{Z}^k , where 1 is at the i -th place, for $1 \leq i \leq k$. Let $\vec{e} = (1, \dots, 1) \in \mathbb{Z}^k$. Then we have the following proposition:

Proposition 4. *Let $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ be a product of projective spaces of dimension $\sum_{i=1}^k n_i = m + 3$, with $m \geq 1$. Fix a configuration matrix*

$$D = \begin{pmatrix} d_1^{(1)} & \cdots & d_k^{(1)} \\ \vdots & & \vdots \\ d_1^{(m)} & \cdots & d_k^{(m)} \end{pmatrix}$$

with $d_j^{(i)} \geq 0 (\forall 1 \leq j \leq k, 1 \leq i \leq m)$ and satisfying the Calabi-Yau condition $\sum_{i=1}^m d_j^{(i)} = n_j + 1 (\forall j = 1, \dots, k)$, then a generic complete intersection Calabi-Yau 3-fold Y in X with configuration matrix D does not contain two smooth rational curve C_1 and C_2 such that:

- The normal bundle $N_{C_i, Y} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, $\forall i = 1, 2$.
- The degree of C_i is in the set $\{\vec{e}_1, \dots, \vec{e}_k, \vec{e}\}$, $\forall i = 1, 2$, and the degrees of C_1 and C_2 are not equal to each other.
- $C_1 \cap C_2 \neq \emptyset$.

Proof. Firstly we collect the coefficients of the homogenous polynomials defining the CICYs to get a parametrization space for CICYs in X .

As before, denote the homogenous coordinates on X as:

$$(X_{10}, \dots, X_{1n_1}; X_{20}, \dots, X_{2n_2}; \dots; X_{k0}, \dots, X_{kn_k})$$

If d_1, \dots, d_k are nonnegative integers, define $V_{(d_1, \dots, d_k)}$ to be the following linear space:

$$\begin{aligned} V_{(d_1, \dots, d_k)} &= \{f(X_{10}, \dots, X_{1n_1}; X_{20}, \dots, X_{2n_2}; \dots; X_{k0}, \dots, X_{kn_k}) \in \mathbb{C}[X_{10}, \dots, X_{kn_k}] : \\ &\quad f \text{ is homogenous with respect to } X_{i0}, \dots, X_{in_i} \text{ with degree } d_i, \forall 1 \leq i \leq k.\} \\ &= H^0(X, \pi_1^* \mathcal{O}_{\mathbb{P}^{n_1}}(d_1) \otimes_{\mathcal{O}_X} \dots \otimes_{\mathcal{O}_X} \pi_k^* \mathcal{O}_{\mathbb{P}^{n_k}}(d_k)) \end{aligned}$$

where $\pi_j : X \rightarrow \mathbb{P}^{n_j}$ is the natural projection, for $1 \leq j \leq k$.

Let $\mathcal{M}_Y = V_{(d_1^{(1)}, \dots, d_k^{(1)})} \times \dots \times V_{(d_1^{(m)}, \dots, d_k^{(m)})}$. Then \mathcal{M}_Y is obviously a parametrization space for CICYs in X with configuration matrix D .

Now we construct a parametrization space for rational curves in X with a fixed degree (d_1, \dots, d_k) , where $d_i \geq 0, \forall 1 \leq i \leq k$. Any such a rational curve is an image of the following morphism:

$$\begin{aligned} \mathbb{P}^1 &\rightarrow X = \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k} \\ (s, t) &\rightarrow (X_{10}(s, t), \dots, X_{1n_1}(s, t); \dots; X_{k0}(s, t), \dots, X_{kn_k}(s, t)) \end{aligned}$$

where $X_{ij}(s, t)$ is a homogenous polynomial of s, t with degree $d_i, \forall 1 \leq i \leq k, 0 \leq j \leq n_i$.

Let $\mathcal{M}_{(d_1, \dots, d_k)}$ be the parametrized rational curves with degree (d_1, \dots, d_k) :

$$\mathcal{M}_{(d_1, \dots, d_k)} = \{f : \mathbb{P}^1 \rightarrow X \mid \text{the degree of } f \text{ is } (d_1, \dots, d_k).\}$$

Clearly $\mathcal{M}_{(d_1, \dots, d_k)}$ is a quasi-projective variety in a natural way. And we let $\mathcal{U}_{(d_1, \dots, d_k)}$ be the nonempty Zariski open subset of $\mathcal{M}_{(d_1, \dots, d_k)}$ whose elements represent smooth rational curves in X with degree (d_1, \dots, d_k) .

If $\{d'_1, \dots, d'_k\}$ is another set of nonnegative integers, let

$$\mathcal{M}_{(d_1, \dots, d_k)}^{(d'_1, \dots, d'_k)} = \{(f_1, f_2) \in \mathcal{U}_{(d_1, \dots, d_k)} \times \mathcal{U}_{(d'_1, \dots, d'_k)} \mid f_1(\mathbb{P}^1) \cap f_2(\mathbb{P}^1) \neq \emptyset\}$$

Then $\mathcal{M}_{(d_1, \dots, d_k)}^{(d'_1, \dots, d'_k)}$ parametrizes intersecting smooth rational curves with degree (d_1, \dots, d_k) and (d'_1, \dots, d'_k) respectively.

Now construct the incident variety:

$$\mathcal{I} = \{(g_1, \dots, g_m, f_1, f_2) \in \mathcal{M}_Y \times \mathcal{M}_{(d_1, \dots, d_k)}^{(d'_1, \dots, d'_k)} : g_i|_{f_1(\mathbb{P}^1)} = g_i|_{f_2(\mathbb{P}^1)} = 0, \forall 1 \leq i \leq m.\}$$

So \mathcal{I} is a parametrization space for the set of tripes (Y, C_1, C_2) , where Y is a CICY in X with configuration matrix D , C_1 and C_2 are smooth rational curves in X with degree (d_1, \dots, d_k) and (d'_1, \dots, d'_k) respectively, $C_1 \cap C_2 \neq \emptyset$, and $C_1 \subset Y, C_2 \subset Y$.

Let $\pi_1 : \mathcal{I} \rightarrow \mathcal{M}_Y$ and $\pi_2 : \mathcal{I} \rightarrow \mathcal{M}_{(d_1, \dots, d_k)}^{(d'_1, \dots, d'_k)}$ be the natural projections. In order to prove the proposition, it suffices to prove $\dim \text{Im} \pi_1 < \dim \mathcal{M}_Y$, where (d_1, \dots, d_k) and (d'_1, \dots, d'_k) are two distinct elements in the set $\{\vec{e}_1, \dots, \vec{e}_k, \vec{e}\}$.

Since for each $Y \in \mathcal{M}_Y$, the fiber $\pi_1^{-1}(Y)$ represents all pairs of parametrized smooth rational curves in Y intersecting each other and with degree (d_1, \dots, d_k) and (d'_1, \dots, d'_k) respectively. Since each smooth rational curve has a reparametrization with dimension $\dim \text{Aut} \mathbb{P}^1 = 3$, so we have the following equality:

$$\dim \text{Im} \pi_1 \leq \dim \mathcal{I} - 6$$

$$\text{Since } \dim \mathcal{I} \leq \dim \mathcal{M}_{(d_1, \dots, d_k)}^{(d'_1, \dots, d'_k)} + \max\{\dim \pi_2^{-1}(p) : p \in \mathcal{M}_{(d_1, \dots, d_k)}^{(d'_1, \dots, d'_k)}\}$$

So it suffices to prove the following inequality:

$$\dim \mathcal{M}_{(d_1, \dots, d_k)}^{(d'_1, \dots, d'_k)} - 6 + \max\{\dim \pi_2^{-1}(p) : p \in \mathcal{M}_{(d_1, \dots, d_k)}^{(d'_1, \dots, d'_k)}\} < \dim \mathcal{M}_Y. \quad (2)$$

Next we will compute explicitly the dimensions above and prove the inequality (2) in each case that $(d_1, \dots, d_k), (d'_1, \dots, d'_k) \in \{\vec{e}_1, \dots, \vec{e}_k, \vec{e}\}$ and $(d_1, \dots, d_k) \neq (d'_1, \dots, d'_k)$.

Consider the case that $(d_1, \dots, d_k) = \vec{e}_1 = (1, 0, \dots, 0), (d'_1, \dots, d'_k) = \vec{e}_2 = (0, 1, 0, \dots, 0)$.

In this case, it is an elementary exercise to show that

$$\dim \mathcal{M}_{(d_1, \dots, d_k)}^{(d'_1, \dots, d'_k)} - 6 = 2n_1 + 2n_2 + n_3 + \dots + n_k - 2$$

Since up to a coordinates changing, we can always assume the pair of curves $(C_1, C_2) \in \mathcal{M}_{(d_1, \dots, d_k)}^{(d'_1, \dots, d'_k)}$ has the following parametrization in the homogenous coordinates of X :

$$C_1 : (s, t, 0, \dots, 0; 1, 0, \dots, 0; 1, 0, \dots, 0; \dots; 1, 0, \dots, 0)$$

$$C_2 : (1, 0, \dots, 0; s, t, 0, \dots, 0; 1, 0, \dots, 0; \dots; 1, 0, \dots, 0)$$

where (s, t) is the homogenous coordinates on \mathbb{P}^1 .

Then we have $\max\{\dim \pi_2^{-1}(p) : p \in \mathcal{M}_{(d_1, \dots, d_k)}^{(d'_1, \dots, d'_k)}\} = \dim \pi_2^{-1}(C_1, C_2)$. And a direct computation shows that:

$$\dim \pi_2^{-1}(C_1, C_2) = \dim \mathcal{M}_Y - \sum_{i=1}^m (d_1^{(i)} + d_2^{(i)} + 1)$$

Note the equality $\sum_{i=1}^k n_i = m + 3$ and the Calabi-Yau condition $\sum_{i=1}^m d_j^{(i)} = n_j + 1 (\forall j = 1, \dots, k)$, putting the computations together, we get the inequality (2), this finishes the proof in the case that $(d_1, \dots, d_k) = \vec{e}_1 = (1, 0, \dots, 0), (d'_1, \dots, d'_k) = \vec{e}_2 = (0, 1, 0, \dots, 0)$.

By the same argument, we get the proof in the case that $(d_1, \dots, d_k) = \vec{e}_i, (d'_1, \dots, d'_k) = \vec{e}_j, i \neq j, 1 \leq i, j \leq k$.

As for the case that $(d_1, \dots, d_k) = \vec{e}_i, (d'_1, \dots, d'_k) = \vec{e}, 1 \leq i \leq k$, the methods are similar. We just give the computations in the case that $(d_1, \dots, d_k) = \vec{e}_1, (d'_1, \dots, d'_k) = \vec{e}$. Other cases are similar.

In the case that $(d_1, \dots, d_k) = \vec{e}_1, (d'_1, \dots, d'_k) = \vec{e}$, it is not difficult to show:

$$\dim \mathcal{M}_{(d_1, \dots, d_k)}^{(d'_1, \dots, d'_k)} - 6 = 2 \sum_{i=1}^k (n_i - 1) + 3(k - 1) + n_1$$

Through a coordinates changing, the pair of curves $(C_1, C_2) \in \mathcal{M}_{(d_1, \dots, d_k)}^{(d'_1, \dots, d'_k)}$ has the following two cases of parametrization in the homogenous coordinates of X .

First case : $C_1 : (s, t, 0, \dots, 0; 1, 0, \dots, 0; \dots; 1, 0, \dots, 0)$

and $C_2 : (s, t, 0, \dots, 0; s, t, 0, \dots, 0; \dots; s, t, 0, \dots, 0)$

Second case : $C_1 : (s, 0, t, 0, \dots, 0; 1, 0, \dots, 0; \dots; 1, 0, \dots, 0)$

and $C_2 : (s, t, 0, \dots, 0; s, t, 0, \dots, 0; \dots; s, t, 0, \dots, 0)$

It can be verified that in each of the two cases,

$$\dim \pi_2^{-1}(C_1, C_2) = \dim \mathcal{M}_Y - \sum_{i=1}^m (2d_1^{(i)} + d_2^{(i)} + \dots + d_k^{(i)} + 1)$$

Note the equality $\sum_{i=1}^k n_i = m + 3$ and the Calabi-Yau condition $\sum_{i=1}^m d_j^{(i)} = n_j + 1 (\forall j = 1, \dots, k)$, putting the computations together, we get the inequality (2), this finishes the proof of the proposition.

□

References

- [1] H. Clemens, *Homological equivalence, modulo algebraic equivalence, is not finitely generated*, Publ. Math. I.H.E.S. 58, 19C38, 1983.

- [2] H. Clemens, *Double solids*, Adv. in Math. 47, 107-230, 1983.
- [3] R. Friedman, *Simultaneous resolution of threefold double points*, Math. Ann. 274(4): 671-689, 1986.
- [4] R. Friedman, *On threefolds with trivial canonical bundle*, Proc. Symp. Pure. Math. 53 (1991).
- [5] P.S.Green, T.Hübsch, *Connetting moduli spaces of Calabi-Yau threefolds*, Comm. Math. Phys. 119: 431C441, 1988.
- [6] S. Katz, *On the finiteness of rational curves on quintic threefolds*, Compos. Math. 60: 151-162, 1986.
- [7] H. P. Kley, *Rigid curves in complete intersection Calabi-Yau threefolds*, Compos. Math. 123: 185-208, 2000.
- [8] A. L. Knutsen, *On isolated smooth curves of low genera in Calabi-Yau complete intersection threefolds*, arXiv:1009.4419v1.
- [9] P.Lu, G.Tian, *Complex Structures on Connected Sums of $S^3 \times S^3$* , Manifolds and geometry (Pisa, 1993), 284-293.
- [10] M. Reid, *The moduli space of 3-folds with $K = 0$ may nevertheless be irreducible*, Math. Ann. 287: 329-334, 1987.
- [11] M. Rossi, *Geometric Transitions*, J. Geom. Phys. 56(9), 1940-1983, 2006.
- [12] G. Tian, *Smoothing 3-folds with trivial canonical bundle and ordinary double points*, Essays on Mirror Manifolds (S.T.Yau ed.), Hongk Kong: International Press, 458-479, 1992.
- [13] A. Vistoli, *The deformation theory of local complete intersections*, arXiv:alg-geom/9703008v2.
- [14] C. Voisin, *On integral Hodge classes on uniruled or Calabi-Yau threefolds*, arXiv:math/0412279v1.
- [15] C.T.C. Wall, *Classification problems in differential topology V. On certain 6-manifolds*, Invent. Math. (1) 355-374, 1966; corrigendum ibid.(2) 306, 1967.